NRL Report 8317

Acoustical Behavior of Thick, Composite, Fluid-Loaded Plates, Calculated Using Timoshenko-Mindlin Plate Theory

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November 16, 1979

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Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
NRL Report 8317

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
ACOUSTICAL BEHAVIOR OF THICK, COMPOSITE, FLUID-LOADED PLATES, CALCULATED USING TIMOSHENKO-MINDLIN PLATE THEORY

5. TYPE OF REPORT & PERIOD COVERED
Interim report on a continuing NRL problem

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Anthony J. Rudgers

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Naval Research Laboratory
Underwater Sound Reference Detachment
P.O. Box 8337, Orlando, FL 32856

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NRL Problem S02-46
Program Element 61153N-11
Project RR011-08-42

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Arlington, VA 22217

12. REPORT DATE
November 16, 1979

13. NUMBER OF PAGES
26

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Acoustical behavior of plates
Acoustic radiation
Acoustic reflection
Composite plates
Homogeneous plates
Fluid-loaded plates
Elastic properties of composite plates
Timoshenko-Mindlin plate theory
Thick-plate theory
Vibration of plates
Structural response function

(Continued)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The reflected and radiated acoustic fields of a thick, composite plate of infinite extent, consisting of a substrate layer bonded at an interface to a coating layer, which has fluid on one side and which is unconstrained on the other side, are calculated theoretically. The reflected and radiated acoustic fields are related by means of a structural response function that completely characterizes the elastic behavior of the composite plate and of the fluid. The composite-plate acoustical problem is analyzed by extending the Timoshenko-Mindlin theory of thick homogeneous plates. The Timoshenko-Mindlin thick-plate equations are used to describe the flexural waves generated in each of the two layers of the

(Continued)

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S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19. Key Words (Continued)

Coupled flexural waves
Welded boundary condition
Perfectly slipping boundary condition
Effective shear modulus
Acoustics of layered media

20. Abstract (Continued)

composite plate, owing either to a plane acoustic wave in the fluid impinging on the plate or to a point-force excitation of the plate. Two ideal types of bonding between the coating and the substrate are considered: the "welded" bond, for which contiguous plate elements on either side of the interface move in complete unison, and the "perfectly slipping" bond, for which such plate elements move in unison normal to the interface but independently parallel to the interface. The analytic form of the structural response function of a composite plate shows that in general a thick bilaminar composite plate of the type considered cannot be modeled by a simple homogeneous thick plate with artificial or "average" material constants. The analysis also shows that the bond between the substrate and the coating significantly affects the acoustic reflection and radiation characteristics of the composite plate.

CONTENTS

ACOUSTICAL BEHAVIOR OF THICK, COMPOSITE, FLUID-LOADED PLATES, CALCULATED USING TIMOSHENKO-MINDLIN PLATE THEORY

INTRODUCTION

The acoustical characteristics of infinitely extended metal plates, loaded on one or both sides by a fluid, have been investigated for a number of years to gain insight into the acoustical behavior of more complicated submerged structures. Although the problems of reflection from homogeneous submerged plates [1] or of the vibration of such plates [2,3] have been solved formally by means of the theory of linear elasticity, it has also proven useful to attack these problems using approximate structural-vibration theories. The approximate theories are useful for several reasons.

First, the results of analyses made using the approximate structural-vibration theories are often much more easily interpreted physically than are those results obtained using exact linear elasticity theory. In fact, when adopting the rigorous elasticity-theory approach, it is often necessary [4] to greatly restrict the scope of one's analytical treatment of a problem that admits a very general formal solution just to cope with the great mathematical complexity of the solution that is finally obtained.

Second, there is little hope of rigorously solving many of the important problems involving sound radiation and reflection by submerged structures by a direct application of linear elasticity theory. Since it will be necessary instead to attack these problems using approximate analysis, it is well to examine for relatively simple structures such as plates the applicability of such approximate structural-vibration theories as will be required in the analysis of complicated structures.

Third, because the approximate structural-vibration theories produce simpler analytical results in general than does linear elasticity theory, the role of the characteristics of structural materials in determining the acoustical behavior of submerged structures can more readily be determined by means of the approximate theories. Since an important objective in structural acoustics is to learn how to modify the acoustic field radiated or reflected by a submerged structure by modifying the characteristics of those materials from which it is fabricated, it is advantageous to employ approximate theories when studying the influence of a particular property of a component material in a structure on the overall acoustical behavior of that structure.

The Timoshenko-Mindlin thick-plate theory is an appropriate structural theory to use [5] in analyzing the acoustic reflection and radiation characteristics of submerged homogeneous plates. (The Timoshenko-Mindlin theory has also been used in evaluating the acoustic transmission through submerged steel plates. See Refs. 6, 7, and 8.) This theory corrects the defects of the classical- or thin-plate theory by accounting for the effects of rotatory inertia and

Manuscript submitted March 15, 1979.

transverse-shear deformation in the motion of the cross-sectional elements of a plate. It has been shown for non-fluid-loaded plates that the results calculated with the Timoshenko-Mindlin theory are indistinguishable from those obtained using linear elasticity theory in the case of the lowest order antisymmetric Lamb mode of a thick plate [9]. (In Ref. 9, Mindlin does not explicitly state that the flexural waves considered in his structural theory are identical to the lowest order antisymmetric Lamb waves. However, it is a straightforward algebraic manipulation to show that the characteristic equation resulting from exact elasticity theory, which Mindlin uses when comparing the exact and the structural plate theories, is the same as the characteristic equation for antisymmetric Lamb waves. An expression for this latter characteristic equation is, for example, Eq. (6-11) of Ref. 3.) According to Junger and Feit [5] the classical plate theory should be abandoned in favor of the Timoshenko-Mindlin thick-plate theory when the plate thickness h is such that

$$2fh/c_s < 0.1 \,, \tag{1}$$

where f is the frequency of the structural waves excited in the plate and c_s is the shear-wave speed in the plate material. (Junger and Feit give this relation as their Eq. (7.6) and cite Ref. 10 as the source, but the relation does not appear in Ref. 10.) One notes from Eq. (1) that the elastic behavior of even relatively thin plates of structural materials having low shear-wave speeds should be analyzed with the Timoshenko-Mindlin theory. For example, a 5-mm-thick plate of a typical rubber, having a nominal shear modulus of elasticity of 13 MN/m^2 and a density of 1.2 Mg/m^3, will satisfy the assumptions of classical plate theory only below a frequency of 1.04 kHz, because the shear-wave speed in the rubber is only 104 m/s.

In this report a bilaminar composite plate of the kind depicted in Fig. 1 will be considered. (Other structural theories that incorporate shear and rotatory-inertia effects have been used to treat the problem of radiation from a submerged bilaminar plate. The results in the present report should be compared to those given in Refs. 11 and 12.) Such a bilaminar plate consists of two thick plates joined at an interface, with a fluid loading one side of the composite and with the other side completely free (vacuum). The two plates in the composite will be referred to as the substrate and the coating. The Timoshenko-Mindlin thick-plate equations will be used to describe the flexural waves excited in both the substrate and the coating. Such flexural waves in the composite plate can be excited in various ways, such as by an acoustic wave in the fluid impinging on the plate or by an applied force driving the substrate. Because the substrate and the coating are joined at an interface, the flexural waves excited in each of the two layers of the composite are coupled.

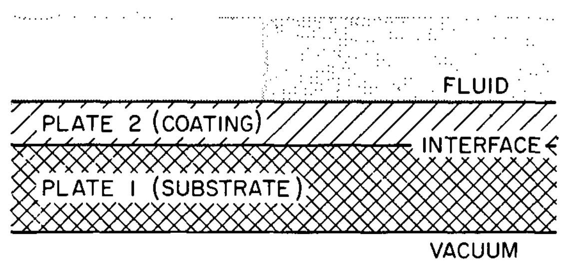

Fig. 1 — A thick fluid-loaded bilaminar composite plate of infinite extent with a free surface. The substrate and the coating in the composite plate are themselves considered as thick plates.

Whether the composite plate is excited by an impinging sound wave or by an applied force, the coupled flexural waves generated will radiate sound into the fluid. In the former situation, one is dealing with the problem of sound reflection by an elastic composite plate, and in the latter situation, one is dealing with a problem of radiation by a fluid-loaded elastic structure. The form of the analytic expression describing the acoustic field of a plate obtained in the reflection problem is different from the form of the expression obtained in the radiation problem. However, it will be shown that the elastic behavior of a plate, whether it be a simple homogeneous plate or a composite plate, can be described by a structural response function that is the same in both the reflection and the radiation problems.

The analytic expression for the structural response function shows that the nature of the bond between the coating and the substrate of the composite influences the acoustical behavior of the bilaminar plate. The nature of the bond determines the boundary conditions to be satisfied at the interface between the coating and the substrate. Thus the nature of the coupling between the flexural waves in the two layers of the bilaminar plate is determined in part by this bond. Two idealized types of such interfacial boundary conditions are considered in this report: "welded" bonding and "perfectly slipping" bonding. Jones and Thrower [13] have previously considered these two types of idealized bonds in their analysis of the vibration of a composite plate with free surfaces and have also given an interesting discussion of the effect of these interfacial boundary conditions on some of the wave-propagation phenomena associated with such a composite plate.

The analysis to be presented here will be developed as follows. Before analyzing the acoustical behavior of a composite plate, a simple uncoated elastic plate will be considered. An expression will be derived that describes the reflection of a harmonic plane acoustic wave from such a plate. This equation will be expressed in terms of the structural response function $\Omega(\omega, \theta)$ of the plate, which is a function of the angular frequency ω of the plane wave that is incident upon the plate at the angle θ. Next the analysis of Feit [14], in the problem of the radiation of sound by an elastic plate excited by a point force, will be considered. It will be shown that the equation describing the farfield directional characteristics of the radiating plate can be expressed in terms of the same structural response function $\Omega(\omega, \theta)$ that appears in the equation describing the acoustic reflection characteristics of the plate. In this radiation problem, ω is the angular frequency of the harmonic point force exciting the plate and θ is the angle of emission of the radiated sound. It will be shown in both the reflection and the radiation problems that the structural response function completely characterizes the plate material and the fluid that overlies it. That is, all of the material properties that enter either problem (plate and fluid densities and elastic moduli), appear only in the quantity Ω.

The analysis of the simple plate shows that the structural response function provides a unique characterization of the material properties of the plate and of the fluid in which it is immersed. This analysis also shows that the structural response function of the plate can be obtained by analyzing either an acoustic reflection or radiation problem. With these facts established, the analysis next turns to the problem of the bilaminar composite plate. It is first shown, in the problem of acoustic reflection by such a plate, that the set of coupled differential equations characterizing the system comprising substrate, coating, and fluid can be transformed into equations that are completely analogous to those obtained in the corresponding problem of the simple uncoated plate. It follows that the only difference between the equation describing the reflected field of the bilaminar plate and the equation describing the reflected field of the simple plate is in the expression that one uses for the structural response function. Thus, without going through any further analysis, the radiated field of the bilaminar plate can be written down directly, simply by substituting the expression obtained for the structural response function of the bilaminar plate in place of that of the simple bare plate in the radiated-pressure equation previously obtained. That is, once the structural response function for a bilaminar composite plate (or for a more complicated plate) is found, the radiated or reflected pressure can be immediately obtained from the equations expressing these latter quantities for the simple bare plate.

REFLECTION OF SOUND BY A THICK FLUID-LOADED PLATE

Consider the infinite elastic plate of thickness h, with fluid above it and vacuum beneath it, that is depicted in Fig. 2. Suppose a harmonic plane acoustic pressure wave in the fluid is incident on the plate at an angle θ. The incident wave of amplitude P_i can be expressed.

$$p_i(\omega, \theta) = P_i \exp[-jk_0(x \sin\theta - z\cos\theta)] \; . \tag{2}$$

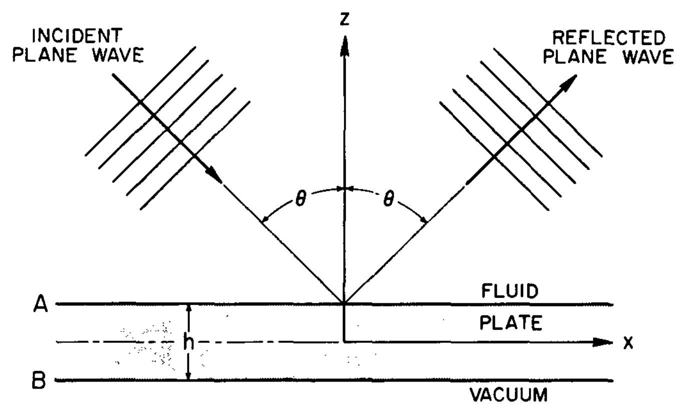

Fig. 2 — Reflection of a plane acoustic wave from an infinitely extended thick elastic fluid-loaded plate with a free surface. The coordinate origin is in the middle surface of the plate.

The reflected pressure wave at an angle θ, equal to the angle of incidence, is then

$$p_r(\omega, \theta) = P_r \exp[-jk_0(x \sin\theta + z\cos\theta)] \; . \tag{3}$$

The factor $\exp(j\omega t)$, with t being the time, has been suppressed in both Eqs. (2) and (3) and will also be suppressed henceforth in this report. In Eqs. (2) and (3), k_0 is the wavenumber of the acoustic wave in the fluid:

$$k_0 = \omega/c, \tag{4}$$

with c being the sound speed in the fluid. The rectangular coordinate system depicted in Fig. 2 has its $z = 0$ coordinate plane at the middle surface of the plate, a distance $h/2$ from either surface. The y axis of the coordinate system is taken to be in the plane of incidence, so that the dependence of quantities on the coordinate y can be eliminated. For convenience the upper and lower surfaces of the plate are labeled A and B respectively. Also of use is a trace wavenumber $\tilde{k}$ defined by

$$\tilde{k} = k_0 \sin\theta. \tag{5}$$

The amplitude P_r of the reflected wave, as a function of the angle θ and of the angular frequency ω of the incident wave, is the quantity to be calculated. This amplitude will depend on the elastic properties of the reflecting plate.

Elastic motion of the plate is described by the Timoshenko-Mindlin equations. For harmonic motion in the coordinate system specified, these equations can be written as

$$D\frac{\partial^2\psi(x)}{\partial x^2} + gh\left[\frac{\partial w(x)}{\partial x} - \psi(x)\right] - \frac{h}{2}[Q_x^A(x) + Q_x^B(x)] + \frac{1}{12}\omega^2\rho_S h^3\psi(x) = 0 \tag{6a}$$

and

$$gh\left[\frac{\partial^2 w(x)}{\partial x^2} - \frac{\partial\psi(x)}{\partial x}\right] + [Q_z^A(x) + Q_z^B(x)] + \omega^2\rho_S h w(x) = 0. \tag{6b}$$

(These Timoshenko-Mindlin equations for a thick plate with general surface stresses can easily be derived by using the same technique that is used in Ref. 5 to derive the Timoshenko-Mindlin equations for a thick plate with normal surface stresses.) The two dependent variables w and ψ in Eqs. (6) are respectively the deflection of the plate in the z direction and the angle describing the effective rotation of the cross-sectional elements of the plate. The quantities Q_z^A and Q_z^B in Eq. (6b) are respectively the normal stresses applied to surfaces A and B of the plate. These stresses are taken to be positive if they are tensile and negative if they are compressive. Similarly Q_x^A and Q_x^B in Eq. (6a) are respectively the applied shear stresses at surfaces A and B. If Q_x^A acts in the positive direction of x, it is taken to be positive, and Q_x^B is taken to be positive if it acts in the direction opposite to the positive direction of x. In the problem being considered, one has for all x

$$Q_z^B(x) = Q_x^B(x) = 0, \tag{7a}$$

since the lower plate surface is stress-free, and

$$Q_x^A(x) = 0, \tag{7b}$$

since the fluid exerts no shearing forces on the plate. At the upper surface A of the plate, where $z = h/2$, one has for the normal stress

$$\begin{aligned} Q_z^A(x) &= -(p_i + p_r)\,|_{z=h/2} \\ &= -(P_0 + P)\exp(-j\tilde{k}x), \end{aligned} \tag{8}$$

with

$$P_0 = P_i \exp[jk_0(h/2)\cos\theta] \tag{9a}$$

and

$$P = P_r \exp[-jk_0(h/2)\cos\theta] \tag{9b}$$

and with $\tilde{k}$ given by Eq. (5).

The other quantities in Eqs. (6) are constants associated with the plate. The flexural rigidity D of the plate is

$$D = \frac{1}{12} Eh^3/(1 - \nu^2), \tag{10}$$

in which E and ν are respectively the Young's modulus of elasticity and the Poisson's ratio of the plate material. The density of the plate material is ρ_S, and g is its effective shear modulus. This latter elastic constant is related to the usual shear modulus of elasticity G of the plate material by the equation

$$g = \kappa^2 G, \tag{11}$$

in which κ is Timoshenko's transverse-shear constant for the plate. Mindlin [9] has shown that for a plate with free surfaces Timoshenko's constant is equal to the ratio of the speed c_R of Rayleigh waves on the surface of a semi-infinite solid consisting of the plate material to the speed c_s of shear waves in that material:

$$\kappa = c_R/c_s, \tag{12}$$

where

$$c_s = \sqrt{G/\rho_S}. \tag{13}$$

The Rayleigh-wave speed c_R is obtained as a root of an appropriate dispersion relation [15].

At the surface A of the plate the velocity must be continuous. Setting the time derivative of the deflection equal to the acoustic particle velocity in the fluid gives

$$w(x) = \frac{1}{\rho\omega^2}\frac{\partial}{\partial z}(p_i + p_r)\,|_{z=h/2}, \tag{14}$$

in which ρ is the density of the fluid. Carrying out the operations indicated in Eq. (14), using the results given in Eqs. (2), (3), (5) and (9), gives

$$w(x) = \frac{jk_0 \cos \theta}{\rho \omega^2} (P_0 - P) \exp(-j\tilde{k}x). \tag{15}$$

Note from the exponential factor in Eq. (15) that the deflection w is expressed as a propagating straight-crested wave. Two coupled differential equations, Eqs. (6), relate the deflection w and the cross-sectional rotation ψ. Hence one also should be able to express ψ, like w, as a propagating straight-crested wave. Therefore the solution for the cross-sectional-rotation wave has the form

$$\psi(x) = C \exp(-j\tilde{k}x), \tag{16}$$

where C is an arbitrary constant.

The amplitude P_r of the reflected pressure wave may now be found. Suppose that

$$w(x) = W \exp(-j\tilde{k}x), \tag{17}$$

where W is also an arbitrary constant. Then substitute this result and that expressed by Eq. (8) into Eq. (6a). The equation thus obtained may be solved for the constant C in terms of W:

$$C = - \frac{j\tilde{k}gh}{D\tilde{k}^2 + gh - \frac{1}{12} \omega^2 \rho_s h^3} W. \tag{18}$$

The results given by Eqs. (8), (16), (17), and (18), when they are substituted into Eq. (6b), yield

$$P_0 + P = \Gamma W, \tag{19}$$

where

$$\Gamma = \omega^2 \rho_s - \tilde{k}^2 gh + \frac{\tilde{k}^2 (gh)^2}{D\tilde{k}^2 + gh - \frac{1}{12} \omega^2 \rho_s h^3} . \tag{20}$$

From Eqs. (15) and (17)

$$P_0 - P = \frac{\rho \omega^2}{jk_0 \cos \theta} W. \tag{21}$$

Eliminating W between Eqs. (19) and (21) and recalling the results expressed by Eqs. (9) gives an expression for the amplitude P_r of the reflected pressure in terms of the amplitude P_i of the incident pressure wave. This expression can be written in the form

$$P_r = P_i \exp(jk_0 h \cos\theta) \left[1 - \frac{2}{1 + j\Omega(\theta, \omega) \cos\theta} \right]. \tag{22}$$

The quantity Ω in Eq. (22) is defined to be the structural response function of the fluid-loaded thick plate. It is given by

$$\Omega(\theta, \omega) = \frac{k_0}{\rho\omega^2} \left[\omega^2 \rho_s h - \tilde{k}^2 gh + \frac{\tilde{k}^2 (gh)^2}{D\tilde{k}^2 + gh - \frac{1}{12}\omega^2 \rho_s h^3} \right]. \tag{23}$$

Note that all the information about the plate material and about the loading fluid (all elastic constants, densities, etc.) that enters the reflection problem is incorporated into the expression for the structural response function of the plate. This information does not appear elsewhere in Eq. (22). The structural response function thus completely characterizes the elastic behavior of the plate material in the problem of reflection of a plane wave by an elastic plate.

The two leading factors on the right-hand side in Eq. (22) are not of great physical significance. The first of these is just the amplitude of the incident wave, and the second is a phase term that arises only because the coordinate origin was taken at the middle surface of the plate. It is therefore useful to also define a normalized reflected pressure

$$\overline{p}_{\text{ref}}(\theta, \omega) = 1 - \frac{2}{1 + j\Omega(\theta, \omega)\cos\theta}. \tag{24}$$

RADIATION OF SOUND BY A THICK FLUID-LOADED PLATE

The analysis outlined here is essentially that given previously by Feit [14]. Here different symbols are used for quantities representing the properties of the plate material, but Feit's basic coordinate systems are retained. Also, to avoid confusion, Feit's convention for the time dependence of a harmonic signal is used, namely, $\exp(-j\omega t)$. The notation in the final expression, which will be obtained in this section of the present report for the radiated field of the plate (Eq. (42)), can be made consistent with the result for the reflected field which was obtained in the previous section (Eq. (22)) by taking the complex conjugate of this final expression and then multiplying the result by a phase factor in order to shift the coordinate origin to the middle surface of the plate.

Consider Fig. 3, which shows an infinite elastic plate of thickness h excited by a harmonic point force applied at the origin of a spherical coordinate system. The polar angle in this system is denoted by θ, the azimuthal angle is denoted by ϕ, and the radial distance from the point of application of the force to the point at which the radiated pressure is observed is denoted by R. An auxiliary cylindrical system, with its coordinates

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(r, z, ϕ) as shown in Fig. 3, is also superimposed on the spherical system. The spherical and cylindrical systems are connected by the relations

$$z = R\cos\theta, \tag{25a}$$

$$r = R\sin\theta, \tag{25b}$$

$$\phi = \phi. \tag{25c}$$

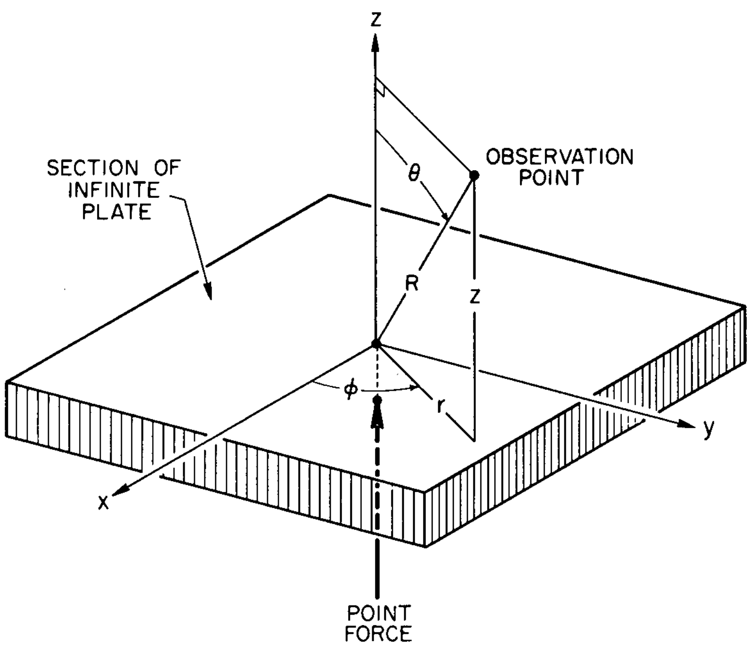

Fig. 3 — Radiation of sound from an infinitely extended thick elastic plate that is excited by a harmonic point force. The plate has fluid above it and vacuum below it. The coordinate origin, which is in the upper surface of the plate (labeled surface A in Fig. 2), is common to the rectangular, the spherical, and the cylindrical systems shown.

As in the reflection problem the plate is considered to have a fluid above it and a vacuum below it. An expression for the radiated pressure field in the fluid at a very great distance from the plate is sought. That is, one wishes to determine the radiated pressure $p(R, \theta, \omega)$ when the point of observation is far enough from the plate so that this pressure can be expressed in the form

$$p(R, \theta, \omega) = P(\theta, \omega)\frac{1}{R}\exp[-j(\omega t - k_0 R)]. \tag{26}$$

One begins with the complete set of three coupled differential equations that, according to the Timoshenko-Mindlin theory, describe the two-dimensional motion of the plate. One then eliminates the two variables ψ_x and ψ_y from these equations. These two variables define the effective rotations of the cross-sectional elements of the plate with respect to x and y rectangular axes, just as ψ defines the rotation of the cross-sectional plate element in the one-dimensional situation described by Eqs. (6). The resultant differential equation describing the deflection of the plate is of the fourth order. In a plane polar coordinate system in the upper surface of the plate, which is described using the coordinates r and ϕ shown in Fig. 3, this fourth-order equation for harmonic motion of the plate is

$$\left[\left(\nabla_1^2 + \frac{\omega^2 \rho_S}{g}\right)\left(D\nabla_1^2 + \frac{1}{12}\rho_S h^3\right) - \rho_S h^3 \omega^2\right] w(r)$$

$$= \left(1 - \frac{D}{gh}\nabla_1^2 - \frac{\omega^2 h^2 \rho_S}{12g}\right)\left[\frac{F_0 \delta(r)}{2\pi r} - p(r, 0, \omega)\right], \quad (27)$$

where

$$\nabla_1^2(\ldots) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right](\ldots). \quad (28)$$

In Eq. (27) the quantity $p(r, 0, \omega)$ is the pressure in the radiated field of the plate expressed in cylindrical coordinates and evaluated at the $z = 0$ plane:

$$p(r, 0, \omega) = p(r, z, \omega)\,|_{z=0} = p(R, \pi/2, \omega). \quad (29)$$

The factor $\delta(r)/2\pi r$ in Eq. (27) is the δ function expressed in polar coordinates, and, as before, $w(r)$ is the deflection of the plate. The quantity F_0 is the amplitude of the driving force. Other quantities appearing in Eq. (27) have been defined previously. By symmetry there is no dependence of Eq. (27) on the coordinate ϕ.

The acoustic field in the fluid satisfies the Helmholtz equation

$$(\nabla^2 + k_0^2)p(r, z, \omega) = 0, \quad (30)$$

with

$$\nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2}, \quad (31)$$

where ∇_1^2 is given by Eq. (28). As before, the velocity must be continuous at the surface of the plate, so that

$$w(r) = \frac{1}{\rho\omega^2}\left[\frac{\partial p(r, z, \omega)}{\partial z}\right]_{|z=0}. \quad (32)$$

The cylindrical symmetry in the problem makes the use of the Hankel transform appropriate. The Hankel-transform pair of zero order, which is used on the coordinate r, is defined by

$$\hat{h}(k,z) = \int_0^\infty h(r,z) J_0(kr) r \, dr \tag{33a}$$

and

$$h(r,z) = \int_0^\infty \hat{h}(k,z) J_0(kr) k \, dk, \tag{33b}$$

where J_0 is the zero-order Bessel function of the first kind and where the caret designates a transformed quantity. When the Hankel transform of Eq. (27) is taken, the result is

$$f(k,\omega)\hat{w}(k) = b(k,\omega)\left[\frac{F_0}{2\pi} - \hat{p}(k,0,\omega)\right], \tag{34}$$

where

$$f(k,\omega) = Dk^4 - \omega^2 \rho_S \left(\frac{D}{g} - \frac{h^3}{12}\right) k^2 - \omega^2 \rho_S h \left(1 - \frac{\omega^2 \rho_S h^2}{12g}\right) \tag{35a}$$

and

$$b(k,\omega) = 1 + \frac{Dk^2}{gh} - \frac{\omega^2 \rho_S h^2}{12g}. \tag{35b}$$

The transformed form of Eq. (30), the Helmholtz equation, is

$$\left[\frac{\partial^2}{\partial z^2} + (k_0^2 - k^2)\right] \hat{p}(k,z,\omega) = 0. \tag{36}$$

The solution of Eq. (36) is

$$\hat{p}(k,z,\omega) = C(k,\omega) \exp[j(k_0^2 - k^2)^{1/2} z], \tag{37}$$

with $\mathrm{Im}[(k_0^2 - k^2)^{1/2} z] > 0$, so that $\hat{p}$ is bounded as $z \to \infty$. In Eq. (37), the quantity C is an arbitrary coefficient that is to be determined by satisfying the transformed form of the boundary condition given by Eq. (32). From this transformed boundary condition and from Eq. (37), one has

$$\hat{w}(k) = \frac{j}{\rho\omega^2} C(k,\omega)(k_0^2 - k^2)^{1/2}. \tag{38}$$

When the results given by Eqs. (37) and (38) are put into Eq. (34), one gets an equation that can be solved for the coefficient C. The expression for this coefficient is

$$C(k,\omega) = \frac{F_0}{2\pi}\left[1 + \frac{j}{\rho\omega^2}(k_0^2 - k^2)^{1/2} \frac{f(k,\omega)}{b(k,\omega)}\right], \tag{39}$$

where f and b are given by Eqs. (35).

The radiated field of plate, owing to excitation by the point force, is obtained by taking the inverse Hankel transform of Eq. (37):

$$p(r, z, \omega) = \int_0^{\infty} C(k, \omega) k J_0(kr) \exp[j(k_0^2 - k^2)^{1/2} z] \, dk. \tag{40}$$

Feit points out how the integral in Eq. (40) can be evaluated. One first transforms this integral into a contour integral in the complex k plane. After reintroducing the original spherical coordinates, using Eqs. (25a) and (25b), one changes the contour of integration by means of the transformation

$$k = k_0 \sin \xi. \tag{41}$$

One then finds that the new contour integral in the complex ξ plane has the form of an integral that has been previously evaluated by Brekhovshikh by means of an asymptotic technique. The asymptotic evaluation of the integral in Eq. (40), which is valid in regions of the fluid where $k_0 R$ is very large (in the farfield of the plate), yields an expression that has the form of Eq. (26), with

$$P(\theta, \omega) = -\frac{jF_0 k_0}{2\pi} \frac{\cos\theta}{1 + \dfrac{jk_0}{\rho\omega^2} \dfrac{f(\widetilde{k}, \omega)}{b(\widetilde{k}, \omega)} \cos\theta}, \tag{42}$$

in which $\widetilde{k}$ is the same quantity as given by Eq. (5).

Suppose that one sets

$$\Omega(\theta, \omega) = -\frac{k_0}{\rho\omega^2} \frac{f(\widetilde{k}, \omega)}{b(\widetilde{k}, \omega)}. \tag{43}$$

It is then a straightforward algebraic manipulation to show that the quantity Ω defined by Eq. (43) is identical to that expressed by Eq. (23). That is, *the structural response function for the radiating plate that is excited by a point force is identical to the structural response function for a reflecting plate that is excited by an impinging plane sound wave.* It is not coincidental, but a consequence of the analysis, that the structural response function of the plate is the same in the reflection and radiation problems. In spherical coordinates the Hankel-transformed equations in the radiation problem, which describe the motion of the plate and the boundary condition at the plate/fluid interface, are formally the same as the corresponding one-dimensional equations that arise in the reflection problem, when one considers the propagation of straight-crested waves in the plate. It can also be shown that, when the result given by Eq. (42) is substituted into Eq. (26), the equation obtained can be transformed into the form of the expression for the radiated field of the plate that is given by Feit. Again one notes that in the radiation problem being considered here the structural response function given by Eq. (43) completely characterizes the elastic behavior of the plate material, just as it did in the reflection problem examined previously.

A normalized form of the radiated pressure, showing the dependence of this quantity on the structural response function of the plate, can be expressed

$$\overline{p}_{\text{rad}}(\theta, \omega) = \frac{\cos\theta}{1 + j\Omega(\theta, \omega)\cos\theta}. \tag{44}$$

In obtaining Eq. (44) from Eq. (26) the complex conjugate of Eq. (42) has been introduced in order to make the harmonic time-dependence factor used in Eq. (44) correspond to that in Eq. (24). Also, if one wishes to change the coordinate origin in the radiation problem from the upper surface of the plate to the middle surface, so as to make the coordinate system used in the radiation problem correspond to the coordinates used in the reflection problem, one should replace the radial distance R in Eq. (26) by the quantity on the right-hand side of the equation:

$$R = R' - \frac{h}{2}\cos\theta. \tag{45}$$

In Eq. (45) the quantity R' is the radial distance to the observation point from the new coordinate origin at the middle surface of the plate. Equation (45), which follows from the cosine theorem for plane triangles, is valid only at distances from the plate such that $R \gg h$. Equation (44), the expression for the normalized radiated pressure, and Eq. (24), the expression for the normalized reflected pressure, should be compared before proceeding. The similarity of the two expressions is to be noted. Also to be noted, as previously mentioned, is that all the material properties of the plate and of the fluid above it are contained entirely in the function $\Omega(\theta, \omega)$ and that Ω is the same in both the reflection and the radiation problems.

STRUCTURAL RESPONSE FUNCTION OF A COMPOSITE PLATE

Calculating the reflected and radiated fields of a bare fluid-loaded plate makes the problem of finding these fields in the case of a bilaminar composite fluid-loaded plate relatively simple. All that is necessary is to find the appropriate expression for the structural response function of the composite plate. This can then be substituted in place of Ω in Eq. (24), if the reflected field of the composite plate is sought, or else substituted in similar fashion into Eq. (44), if the radiated field of the composite plate is to be found. Moreover the analysis needed to obtain the structural response function for a composite plate closely parallels either the development that was used previously in the analysis of the reflection from the simple bare plate or else that used in determining the farfield radiation from such a plate. Since it does not matter whether a reflection or a radiation problem is analyzed in order to calculate a structural response function, the reflection problem will be chosen in the case of the composite plate, because the analysis is simpler in this problem than in the radiation problem.

"Welded" Interfacial Bond

Consider Fig. 4, which is essentially the same as Fig. 1, but with two rectangular coordinate systems, an unprimed and a primed system, superimposed on it. The origin of the unprimed system is at the middle surface of the substrate, and the origin of the primed system is at the middle surface of the coating. In the discussion to follow, unprimed quantities will generally be used in connection with the substrate, and primed quantities will be used in connection with the coating. For example, h and h' refer to the thicknesses of the substrate and the coating respectively and ρ_S and ρ'_S refer to the respective densities. The upper and lower surfaces of the substrate are labeled A and B, and the upper and lower surfaces of the coating are labeled A' and B', as shown in Fig. 4, with A and B' coincident.

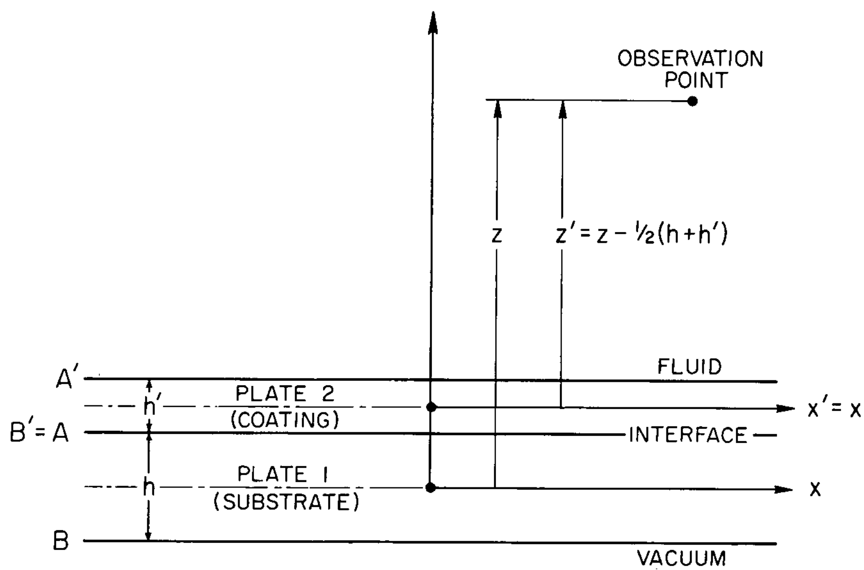

Fig. 4 — Geometry of a bilaminar composite plate. Two rectangular coordinate systems are used. Each system has its origin in the middle surface of one of the two thick plates composing the composite. In the text, unprimed quantities in general refer to plate 1, the substrate, and primed quantities refer to plate 2, the coating.

The motion of the substrate is described by Eqs. (6). The motion of the coating is described by a similar pair of equations:

$$D' \frac{\partial^2 \psi'}{\partial x^2} + g'h'\left(\frac{\partial w'}{\partial x} - \psi'\right) - \frac{h'}{2}\left({Q_x^A}' + {Q_x^B}'\right) + \frac{1}{12}\omega^2 \rho'_S (h')^3 \psi' = 0 \tag{46a}$$

and

$$g'h'\left(\frac{\partial^2 w'}{\partial x^2} - \frac{\partial \psi'}{\partial x}\right) + \left({Q_z^A}' + {Q_z^B}'\right) + \omega^2 \rho'_S h' w' = 0. \tag{46b}$$

As before, the superscripts on the stress terms Q designate the surface on which the stresses act, and the subscripts designate the direction of their action. A number of the boundary conditions in the present problem are like those encountered in the case of the bare plate. Thus one has

$$Q_x^{A'} = Q_x^B = Q_z^B = 0, \tag{47}$$

and

$$Q_z^{A'} = -(p_\mathrm{i} + p_\mathrm{r})\,|_{z=(h/2)+h'} = -(P_0 + P)\exp(-j\tilde{k}x), \tag{48}$$

with

$$P_0 = P_\mathrm{i}\exp(jk_0\gamma) \tag{49a}$$

and

$$P = P_\mathrm{r}\exp(-jk_0\gamma), \tag{49b}$$

where

$$\gamma = \left(\frac{h}{2} + h'\right)\cos\theta. \tag{50}$$

Since the normal displacements of the coating and substrate must be the same, one has

$$w(x) = w'(x) \tag{51}$$

for all values of x. As before, the velocity of the plate must match the particle velocity in the fluid at the upper surface of the plate. If one applies Eq. (14) at $z = (h/2) + h'$, an expression identical in form to Eq. (15) is obtained, but with P_0 and P now given by Eqs. (49). From this expression one obtains, as in the problem of the bare plate, an equation with exactly the form of Eq. (21).

Next consider the boundary conditions at the interface $A = B'$ between the substrate and the coating. In the case of the welded interface being considered, a pair of contiguous differential plate elements on opposite sides of the interface move in complete unison. The forces exerted by such a differential element in the coating on the adjacent element in the substrate must be equal and opposite to those exerted by the element in the substrate on the element in the coating. Thus, since the normal and shearing stresses are in equilibrium across the interface, one has for all x

$$Q_z^A(x) = -Q_z^{B'}(x) \equiv Q_z(x) \tag{52a}$$

and

$$Q_x^A(x) = -Q_x^{B'}(x) \equiv Q_x(x). \tag{52b}$$

The quantities Q_z and Q_x defined by Eqs. (52) are new designations for the interfacial stresses. One additional boundary condition must also be satisfied; the tangential displacements $u(x)$ and $u'(x)$ in the substrate and coating must be equal at an interface with a welded bond. Since, in the coordinate systems used, one has

$$u(x) = z\psi(x) \tag{53a}$$

and

$$u'(x) = z'\psi'(x), \tag{53b}$$

the boundary condition

$$u'(x)\,|_{z'=-h'/2} = u(x)\,|_{z=h/2} \tag{54}$$

leads to

$$\psi'(x) = -\beta\psi(x), \tag{55}$$

where

$$\beta = h/h'. \tag{56}$$

Equations (51) and (55) allow the variables w' and ψ' to be eliminated from Eqs. (6) and (46).

As in the case of the bare plate, one considers straight-crested harmonic waves propagating along the composite plate. Thus one expresses ψ and w by Eqs. (16) and (17) respectively and also takes

$$Q_x(x) = q_x \exp(-j\widetilde{k}x) \tag{57a}$$

and

$$Q_z(x) = q_z \exp(-j\widetilde{k}x). \tag{57b}$$

When the results expressed by Eqs. (16), (17), (47), (48), (49), (51), (52), (55), and (57) are all substituted into Eqs. (6) and (46), four coupled algebraic equations result:

$$-D\widetilde{k}^2C - gh(j\widetilde{k}W + C) - \frac{h}{2}q_x + \frac{1}{12}\,\omega^2\rho_S' h^3 C = 0, \tag{58a}$$

$$\beta D'\widetilde{k}^2C - g'h'(j\widetilde{k}W - \beta C) + \frac{h'}{2}q_x - \frac{1}{12}\,\omega^2\rho_S'(h')^3\beta C = 0, \tag{58b}$$

$$gh(-\widetilde{k}^2W + j\widetilde{k}C) + q_z + \omega^2\rho_S hW = 0, \tag{59a}$$

and

$$g'h'(-\tilde{k}^2 W - j\beta\tilde{k}C) - q_z - (P_0 + P) + \omega^2\rho_S' h' W = 0. \tag{59b}$$

Suppose that Eq. (58a) is multiplied by h', that Eq. (58b) is multiplied by h, and that the two equations thus obtained are added together to eliminate q_x. If the resulting equation is then solved for the coefficient C, the expression obtained is

$$C = -\frac{j\tilde{k}(g + g')h}{(D - \beta^2 D')\tilde{k}^2 + (g - \beta g')h - \dfrac{\omega^2}{12}(\beta\rho_S - \rho_S')h^2 h'} W. \tag{60}$$

Next Eqs. (59a) and (59b) are added together to eliminate q_z, and the resulting equation is solved for $P_0 + P$. The expression obtained is

$$P_0 + P = [\omega^2(\rho_S h + \rho_S' h') - \tilde{k}^2(gh + g'h')]W + j\tilde{k}h(g - g')C. \tag{61}$$

When the value of the coefficient C, given by Eq. (60), is substituted into Eq. (61), the result can be put in exactly the form of Eq. (19). On dividing the two expressions, which correspond to Eqs. (19) and (21) in the bare-plate case, and solving the resulting equation for the reflected pressure, one gets

$$P_r = P_i \exp(j2k_0\gamma)\left[1 - \frac{2}{1 + j\Omega(\theta, \omega)\cos\theta}\right], \tag{62}$$

with γ given by Eq. (50). Equation (62) is, as expected, of the same form as Eq. (22) which was obtained in the bare-plate reflection problem. Here, however, for the bilaminar plate with a welded bond at the interface, the structural response function Ω is expressed

$$\Omega(\theta, \omega) = \frac{k_0}{\rho\omega^2}\left[\omega^2(\rho_S h + \rho_S' h') - \tilde{k}^2(gh + g'h') + \frac{\tilde{k}^2[g^2 - (g')^2]^{1/2} h}{(D - \beta^2 D')\tilde{k}^2 + (g - \beta g')h - \dfrac{1}{12}\omega^2(\beta\rho_S - \rho_S')h^2 h'}\right]. \tag{63}$$

Obviously Eq. (62), which gives the reflected field of the composite plate, may be written in the normalized form of Eq. (24). Also, as stated previously, one simply substitutes the result given by Eq. (63) into Eq. (44) to obtain the normalized radiated field of the composite plate, owing to excitation by a point force.

"Perfectly Slipping" Interfacial Bond

The problem of a "perfectly slipping" interface is similar to the problem of the welded interface, and Fig. 4 may again be used. The motions of the substrate and coating are again described by Eqs. (6) and (46) respectively. Moreover the boundary conditions specified by Eqs. (47) and (48) also hold in this situation.

For a "perfectly slipping" condition at the interface $A = B'$, normal motions of the coating and substrate will be directly coupled together, but tangential motions will not. That is, a pair of contiguous differential elements on opposite sides of the interface move in complete unison in the direction perpendicular to the interface but move completely independently in the direction parallel to it. This type of motion implies that the boundary conditions given by Eqs. (51) and (52a) also hold in the present situation. Equation (52b), however, must be replaced by

$$Q_x^A = Q_x^B = 0, \tag{64}$$

since the coating and substrate exert no tangential forces on one another. Equation (55) also no longer holds, since there is no direct interfacial coupling between the tangential displacements in the two layers of the composite plate. Therefore, in addition to specifying ψ by Eq. (16), one likewise sets

$$\psi'(x) = C' \exp(-j\widetilde{k}x), \tag{65}$$

with C' an arbitray constant.

When the results expressed by Eqs. (16), (17), (47), (48), (49), (51), (52), (57), (64), and (65) are all substituted into Eqs. (6) and (46), four coupled algebraic equations again result:

$$-D\widetilde{k}^2C - gh(j\widetilde{k}W + C) + \frac{1}{12}\omega^2\rho_S h^3 C = 0, \tag{66a}$$

$$-D'\widetilde{k}^2C' - g'h'(j\widetilde{k}W + C') + \frac{1}{12}\omega^2\rho_S'(h')^3 C' = 0, \tag{66b}$$

$$gh(-\widetilde{k}^2W + j\widetilde{k}C) + q_z + \omega^2\rho_S hW = 0, \tag{67a}$$

and

$$g'h'(-\widetilde{k}^2W + j\widetilde{k}C') - q_z - (P_0 + P) + \omega^2\rho_S'h'W = 0. \tag{67b}$$

Equation (66a) and Eq. (66b) are solved respectively for C and C' in terms of W, giving

$$C = -\frac{j\widetilde{k}gh}{D\widetilde{k}^2 + gh - \frac{1}{12}\omega^2\rho_S h^3}\,W \tag{68a}$$

and

$$C' = -\frac{j\widetilde{k}g'h'}{D'\widetilde{k}^2 + g'h' - \frac{1}{12}\omega^2\rho_S'(h')^3}\,W. \tag{68b}$$

Next Eqs. (67a) and (67b) are added together to eliminate q_z, and the resulting equation is solved for the quantity $P_0 + P$ as before. The result is

$$P_0 + P = [\omega^2(\rho_S h + \rho_S'h') - \widetilde{k}^2(gh + g'h')]\,W + j\widetilde{k}ghC + j\widetilde{k}g'h'C'. \tag{69}$$

After substituting the expressions for C and C', given by Eqs. (68), into Eq. (69), one can eliminate W and solve the resulting equation for the amplitude P_r of the reflected pressure in terms of the amplitude P_i of the incident wave. This calculation, which is identical to that performed in the case of the welded interface, yields again Eq. (62). In the present case of the composite plate with a perfectly slipping interface, however, the structural response function is

$$\Omega(\theta, \omega) = \frac{k_0}{\rho\omega^2}\left[\omega^2(\rho_s h + \rho_s' h') - \tilde{k}^2(gh + g'h') + \frac{(\tilde{k}gh)^2}{D\tilde{k}^2 + gh - \frac{1}{12}\omega^2\rho_s h^3} + \frac{(\tilde{k}g'h')^2}{D'\tilde{k}^2 + g'h' - \frac{1}{12}\omega^2\rho_s'(h')^3}\right]. \qquad (70)$$

USING THE STRUCTURAL RESPONSE FUNCTION TO DESCRIBE THE ACOUSTICAL BEHAVIOR OF PLATES

The structural response function of a bare plate is given by Eq. (23), that for a composite plate with a welded interfacial bond is given by Eq. (63), and that for a composite plate with a perfectly slipping interfacial bond is given by Eq. (70). A number of results pertaining to the acoustical behavior of plates can be obtained by examining these equations and by using the general concept of structural response functions.

First, neither of the two structural response functions found for a composite plate can be put in the exact form of the structural response function of the simple bare plate by specifying a set of artificial or "effective" material constants for the composite. That is, it is impossible to determine a set of effective constants h^*, ρ_s^*, g^*, and D^* such that if each of these constants were taken to be some combination of the actual material constants h, ρ_s, g, and D and h', ρ_s', g', and D', then Eq. (63) or Eq. (70) would be reduced to the form of Eq. (23). This impossibility can easily be seen by means of an example. By specifying the effective values g^* and h^* such that

$$g^*h^* = gh + g'h', \qquad (71)$$

the second term in brackets in Eq. (63) could be put in the same form as the second term in brackets in Eq. (23). However, to make the second term in the denominator of the last factor in Eq. (63) have the form of its counterpart in Eq. (23), one would be required to take

$$g^*h^* = gh - g'h^2/h'. \qquad (72)$$

Moreover, to make the numerator of the last factor in both Eq. (63) and Eq. (23) have the same form, it would be necessary to set

$$g^*h^* = [g^2 - (g')^2]^{1/2}h. \qquad (73)$$

Clearly one cannot find a pair of effective material constants g^* and h^* that satisfies Eqs. (71), (72), and (73) simultaneously. Similar problems also occur when attempting to assign effective values to other material constants. Thus thick-plate theory shows that there is no analytic basis for treating a composite layered material as a homogeneous material with artificial, "effective" material constants—a practice often encountered in thin-plate theory and shell theory. One could possibly find such a set of effective material constants, however, if he were willing to greatly restrict the frequency range and the range of angles of observation being considered. Alternatively, by allowing the values of the effective constants to vary rather arbitrarily with both frequency ω and angle θ, one conceivably could also find such a set of effective constants. However, little physical significance could be attached to the effective material constants found by either of these two procedures.

Second, the analytic forms of Eqs. (63) and (70) are significantly different. Thus, in general, the nature of the bond between the coating and the substrate layers of a composite plate must affect the acoustical behavior of the overall structure. The implications of this fact need to be examined in practical applications where composite structures are employed to control acoustic radiation and reflection. Generally, when adhesives are used in practical situations to bond layers of a composite structure together, acoustical considerations play no role at all when a designer selects the bonding materials.

Third, the acoustical behavior of the composite plate is independent of which of the two layers is vacuum backed and which is fluid loaded. That is, for a substrate with thickness h and a coating of another material with thickness h', the structural response function of the composite plate when the coating is in contact with the fluid and the substrate is vacuum-backed is the same as the structural response function when the substrate is in contact with the fluid and the coating is vacuum backed. This result can be deduced by interchanging the primed and the unprimed quantities relating to the composite in either Eq. (63) or Eq. (70). Because of this symmetry in the structural response function, neither the reflected pressure, owing to excitation of the plate by an incident plane wave, nor the radiated pressure, owing to excitation of the plate by a point force, is altered if the composite plate is reversed. This fact is a consequence of using the Timoshenko-Mindlin thick-plate equations to describe the motion of the plate.

Fourth, the composite plate is always perfectly reflecting if there are no losses in either the coating or the substrate. That is, if the materials in both the substrate and the coating are perfectly elastic, then the magnitude of the reflected pressure is independent of the particular constants that describe these materials and is independent of the thicknesses of the coating and the substrate. This independence holds true at all frequencies and at all angles of incidence of the impinging plane wave. To demonstrate this result, one has only to calculate the modulus of the normalized reflected pressure $\overline{p}_{\mathrm{ref}}$, given by Eq. (24). From this calculation one finds that the modulus of the complex $\overline{p}_{\mathrm{ref}}$ is

$$|\overline{p}_{\mathrm{ref}}(\theta, \omega)| = 1, \tag{74}$$

provided that $\Omega(\theta, \omega)$ is a real quantity. Now $\Omega(\theta, \omega)$ will be real if the effective shear moduli g and g' and the flexural rigidities D and D' of the substrate and coating layers are real quantities, since all other parameters in the expressions for the structural response functions can have only real values. Recalling Eqs. (10) and (11), one sees that g and

D will be real quantities if the several elastic constants G, E, and ν, of the substrate material are all real quantities, that is, if the substrate material is perfectly elastic. A similar argument applies to the constants g' and D' of the coating. Also, since Eq. (74) results from the form of Eq. (24), the simple bare plate, like the composite plate, is likewise a perfect reflector at all frequencies and for all incidence angles, provided that it is perfectly elastic. The result expressed by Eq. (74) is somewhat unexpected. According to the analysis presented, the only effect that the properties of a perfectly elastic material have on the acoustic reflection characteristics of a vacuum-backed plate made from that material is to alter the phase of the reflected plane wave relative to that of the incident plane wave. It might be argued that Eq. (74) is an obvious result, since one is dealing with a lossless system. In such a case, the argument would run, all acoustic energy that strikes the plate must subsequently be reflected from it. This naive argument, however, should not be applied in the problem analyzed for the following reason. An elastic plate, whether it be simple or composite, will convert some of the incident acoustic energy into elastic energy and, by its vibration, will store part of this elastic energy. The amount of the incident acoustic energy stored as elastic energy will certainly depend not only on the frequency and angle of incident of the impinging plane wave but also on the elastic parameters of the material. The surprising feature about this conversion and storage of energy is that it occurs in just such a way that Eq. (24), which leads to Eq. (74), holds true.

Finally, from the appropriate structural response function the coincidence frequency ω_c of a composite or of a simple bare plate may be readily derived by using the equation

$$\Omega(\pi/2, \omega_c) = 0. \tag{75}$$

To illustrate the use of Eq. (75), suppose that it is applied in the case of the simple bare plate. One then has from Eqs. (4), (5), (23), and (75)

$$\left(D\omega_c^2 + c^2 gh - \frac{1}{12}\omega_c^2 c^2 \rho_S h^3\right)(c^2\rho_S - g) + c^2 g^2 h = 0. \tag{76}$$

When Eq. (76) is solved for ω_c^2, the result obtained is

$$\omega_c^2 = \frac{c^4 \rho_S h}{D\left(1 - \dfrac{c^2\rho_S}{g}\right)\left(1 - \dfrac{c^2 \rho_S h^3}{12D}\right)}. \tag{77}$$

This expression for the coincidence frequency of a Timoshenko-Mindlin plate is the same as Feit [14] gave in his List of Symbols. Moreover, for a simple or a composite plate, a coincidence angle θ_c will exist at a given frequency ω if

$$\Omega(\theta_c, \omega) = 0 \tag{78}$$

has a real solution for θ_c. Illustrating the use of Eq. (78) by again considering only the simple bare plate, one has from Eqs. (4), (5), (23), and (78)

$$\left(D\omega^2 \sin^2\theta_c + c^2 gh - \frac{1}{12}\omega^2 \rho_S c^2 h^3\right)(c^2\rho_S - g) + g^2 hc^2 = 0. \tag{79}$$

If Eq. (79) is solved for $\sin^2 \theta_c$, the result is

$$\sin^2 \theta_c = \frac{c^2 \rho_S h^3}{12D} \left[1 + \frac{12c^2}{\omega^2 h^2 \left(1 - \frac{c^2 \rho_S}{g}\right)} \right] . \tag{80}$$

The coincidence angle θ_c is obtained by taking the inverse sine of the square root of the right-hand side of Eq. (80). This operation will yield a real value of θ_c; that is, a coincidence angle will exist only if the right-hand side does not exceed unity. For a given set of plate parameters the right-hand side of Eq. (80) will be less than unity when the frequency ω of the harmonic excitation of the plate is equal to or greater than the coincidence frequency ω_c given by Eq. (77).

SUMMARY

A unified analytical treatment has been given of the problems of reflection and radiation from a thick composite plate that is fluid-loaded on one side and has a vacuum on the other. The radiation and reflection problems have been related by introducing the concept of the structural response function of the plate. By use of this function it has been shown that one can characterize the elastic behavior of a thick plate that may be either simple or composite. It has been found that the structural response function of a plate contains all of the information about the plate materials that enters either the reflection or the radiation problem. Expressions for the structural response function for two types of composite bilaminar plates have been derived. It has been shown that, because these functions are of an analytic form different from the structural response function of a simple plate, one cannot in general determine a set of average or "effective" material constants that make a thick bilaminar plate equivalent to a homogeneous thick plate. Also for the two types of bilaminar plates considered, it has been shown, that, because the structural response functions are different, the nature of the bond joining the layers of a bilaminar plate must play a role in determining the overall acoustical behavior of the bilaminar plate structure. Moreover it has also been demonstrated that the structural response function is a concept that is quite useful when describing a number of the acoustical phenomena that occur in radiation or reflection of sound by thick fluid-loaded plates.

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